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# ON AN ENCOUNTER-EVASION DIFFERENTIAL GAME

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We investigate the conditions for the solvability of a differential game, based on a program construction analogous to [1, 2]. We quote the conditions for the existence of the equilibrium situation in pure strategies. The paper abuts the investigations in [1 – 8].

**1.** Consider the conflict-controlled system

$$dx/dt = f(t, x, u, v), \qquad x(t_0) = x_0$$
$$x \in R^n, \quad u \in P \subset R^p, \quad v \in Q \subset R^q$$

Here $f(\cdot)$ is a function continuous in all arguments and continuously differentiable in $x$, satisfying the condition for uniform continuability of solutions formulated in [3, 7, 8], $P$ and $Q$ are the first and second player's compact sets of admissible controls.

A closed set $\Theta$ is delineated on the interval $[t_0, \vartheta_0]$. We assume that the function $\omega(\vartheta, x, m)$ is given on the set $\{(\vartheta, x, m) : (\vartheta, m) \in M, x \in R^n\}$, where $M$ is a compact subset of $\Theta \times R^m$, and $\omega(\cdot)$ is continuous in all arguments and continuously differentiable in $x$ in the region $\omega_0 < \omega < \omega^\circ$. Without loss of generality we assume that the sections

$$M_\vartheta = \{m : (\vartheta, m) \in M, \ m \in R^m\}$$

are not empty for all $\vartheta \in \Theta$ and $\max_\Theta \vartheta = \vartheta_0$.

We assume that the strategies $U$ and $V$, the counterstrategy $U_v$, and the motions generated by them are defined analogously to [8] by passing to a limit from the corresponding Euler polygonal lines.

P r o b l e m 1. Construct a strategy $U^\circ$ or a counterstrategy $U_v^\circ$ which on any motion $x_{U^\circ}[t]$ and, respectively, $x_{U_v^\circ}[t]$ guarantees the fulfillment of the inequality

$$\min_\Theta \min_{M_\vartheta} \omega(\vartheta, x_{U^\circ}[\vartheta], m) \leqslant \varepsilon \tag{1.1}$$

$$\min_\Theta \min_{M_\vartheta} \omega(\vartheta, x_{U_v^\circ}[\vartheta], m) \leqslant \varepsilon \tag{1.2}$$

where $\varepsilon$ is a preassigned number.

P r o b l e m 2. Construct the pair of strategies $(U^\circ, V^\circ)$ for which the inequality

$$\sup_{\{x_{U^\circ,V}[t]\}} \min_\Theta \min_{M_\vartheta} \omega(\vartheta, x_{U^\circ,V}[\vartheta], m) \leqslant$$
$$\min_\Theta \min_{M_\vartheta} \omega(\vartheta, x^\circ[\vartheta], m) \leqslant$$
$$\inf_{\{x_{U,V^\circ}[t]\}} \min_\Theta \min_{M_\vartheta} \omega(\vartheta, x_{U,V^\circ}[\vartheta], m)$$

is fulfilled on every motion $x^\circ[t] = x_{U^\circ, V^\circ}[t]$ whatever be the strategies $U$, $V$.

Problem 3. Construct the strategy $V^\circ$ guaranteeing the inequality

$$\min_\theta \min_{M_\vartheta} \omega(\vartheta, x_{V^\circ}[\vartheta], m) \geqslant \varepsilon$$

on any motion $x_{V^\circ}[t]$ ; ($\varepsilon$ is a given number).

**2.** Let us consider a modification of the program construction in [7, 8]. Let $\{H(m(\cdot)), [t_*, \vartheta]\}$ be the class of admissible program controls $\eta(\cdot)$, $\{K(m(\cdot)), [t_*, \vartheta]\}$ be the class of the first player's program controls $\mu(\cdot)$, $\{E(m(\cdot)), [t_*, \vartheta]\}$ be the class of the second player's program controls $\nu(\cdot)$, identified, respectively, with the collections of all regular Borel measures on the products $[t_*, \vartheta] \times P \times Q$, $[t_*, \vartheta] \times P$ and $[t_*, \vartheta] \times Q$, having Lebesgue projection on $[t_*, \vartheta]$ [7, 8]. Let $\sigma_{[t_*, \vartheta]}$ be the $\sigma$-algebra of Borel subsets on $[t_*, \vartheta]$. Then for every measure $\eta(\cdot) \in \{H(m(\cdot)), [t_*, \vartheta]\}$ there exists a function $\eta_t(\cdot)$, unique to within values on a set of Lebesgue measure zero, named below the instantaneous program control, whose values for each $t \in [t_*, \vartheta]$ are probabilities on $P \times Q$; moreover, for every Borel subset $K \subset P \times Q$ the function $\eta_t(K)$ $\sigma_{[t_*, \vartheta]}$ is measurable and

$$\eta(\{(t,u,v): t \in \Gamma, (u,v) \in K\}) = \int_\Gamma \eta_t(K) m(dt)$$

for any Borel subsets $\Gamma \subset [t_*, \vartheta]$ and $K \subset P \times Q$. Analogously we define the first and second players' instantaneous program controls $\mu_t(\cdot)$ and $\nu_t(\cdot)$, corresponding to the measures $\mu(\cdot) \in \{K(m(\cdot)), [t_*, \vartheta]\}$ and $\nu(\cdot) \in \{E(m(\cdot)), [t_*, \vartheta]\}$, respectively.

For an arbitrary $\sigma_{[t_*, \vartheta]}$-measurable function $u(\cdot)$ we denote by $\delta_{u(t)}$ the instantaneous program control $\mu_t(\cdot)$ concentrated at the point $u_t = u(t)$ for each $t$. The notation $\delta_{v(t)}$ has an analogous meaning. Let

$$\{K^*(m(\cdot)), [t_*, \vartheta]\}, \quad \{E^*(m(\cdot)), [t_*, \vartheta]\}$$

be subclasses of $\{K(m(\cdot)), [t_*, \vartheta]\}$ and $\{E(m(\cdot)), [t_*, \vartheta]\}$, consisting, respectively, of all such controls $\mu^*(\cdot)$ and $\nu^*(\cdot)$ that the instantaneous controls $\mu_t^*(\cdot)$ and $\nu_t^*(\cdot)$ corresponding to them are $\delta_{u^*(t)}$ and $\delta_{v^*(t)}$, respectively, where $u^*(t) \in P$, $v^*(t) \in Q$ are $\sigma_{[t_*, \vartheta]}$-measurable vector-valued functions. By the weak convergence of the program controls $\eta(\cdot)$, $\mu(\cdot)$ and $\nu(\cdot)$ we mean their convergence in the $*$-weak topology of the spaces adjoint to $C([t_*, \vartheta] \times P \times Q)$, $C([t_*, \vartheta] \times P)$ and $C([t_*, \vartheta] \times Q)$, respectively. The following lemma can be proved by using the results in [7].

Lemma 2.1. The sets $\{K^*(m(\cdot)), [t_*, \vartheta]\}$ and $\{E^*(m(\cdot)), [t_*, \vartheta]\}$ are weakly dense in $\{K(m(\cdot)), [t_*, \vartheta]\}$ and $\{E(m(\cdot), [t_*, \vartheta]\}$, respectively.

With an arbitrary position $(t_*, x_*)$, $t_* \in [t_0, \vartheta_0]$ we associate the quantity

$$\varepsilon_0(t_*, x_*) = \max_{\{E(m(\cdot)), [t_*, \vartheta_0]\}} \min_{X(\cdot, t_*, x_*, \nu(\cdot))} \min_{\Theta_{t_*}} \min_{M_\vartheta} \omega(\vartheta, x(\vartheta), m) = \max_{\{E(m(\cdot)), [t_*, \vartheta_0]\}} \rho_M(X(\cdot, t_*, x_*, \nu(\cdot))) \tag{2.1}$$

where $X(\cdot, t_*, x_*, \nu(\cdot))$ is the sheaf of all program attainments [3, 7, 8] generated by the program $\{\Pi(\nu(\cdot)), [t_*, \vartheta_0]\}$ [7, 8], $\Theta_{t_*} = \Theta \cap [t_*, \vartheta_0]$. We emphasize that the corresponding maxima and minima in (2.1) are actually achieved, which follows from the weak compactness in itself of the programs of class $\{E(m(\cdot)), [t_*, \vartheta_0]\}$, as well as from the results in [7]. Allowing for Lemma 2.1, we can show that

$$\varepsilon_0(t_*, x_*) = \sup_{\{v(\cdot)\}} \inf_{\{u(\cdot)\}} \min_{\Theta_{t_*}} \min_{M_\vartheta} \omega(\vartheta, \varphi(\vartheta, t_*, x_*, u(\cdot), v(\cdot)), m)$$

where $\{u(\cdot)\}$ and $\{v(\cdot)\}$ are collections of all $\sigma_{[t_*, \vartheta]}$-measurable functions, $\varphi(t, t_*, x_*, u(\cdot), v(\cdot))$ is the solution of the differential equation

$$dx/dt = f(t, x, u(t), v(t)), \quad x(t_*) = x_*$$

We note that in the expression for $\varepsilon_0(\cdot)$ the sets $\{u(\cdot)\}$ and $\{v(\cdot)\}$ can also be assumed to be the sets of all piecewise-constant vector-valued functions with values in $P$ and $Q$, respectively. We can define the quantity $\rho_M(X(\cdot, t_*, x_*, \nu(\cdot)))$ occurring in (2.1) also in terms of the attainability region [1] $G(\vartheta, t_*, x_*, \nu(\cdot))$ for the program $\{\Pi(\nu(\cdot)), [t_*, \vartheta_0]\}$ in the following way:

$$\rho_M(X(\cdot, t_*, x_*, \nu(\cdot)) = \min_{\Theta_{t_*}} \min_{G(\vartheta, t_*, x_*, \nu(\cdot))} \min_{M_\vartheta} \omega(\vartheta, x, m)$$

By $\sigma(t_*, x_*)$ we denote the set of all optimal program controls of the second player, which yield the maximum in (2.1), and by $X^\circ(\cdot, t_*, x_*, \nu(\cdot))$ and $\{\Pi(\nu(\cdot)), [t_*, \vartheta_0] \mid t_*, x_*\}_0$ we denote the set of all program motions optimal in the sheaf $X(\cdot, t_*, x_*, \nu(\cdot))$ [2, 3, 7, 8] and the set of optimal controls from the program $\{\Pi(\nu(\cdot)), [t_*, \vartheta_0]\}$, respectively: for each $x^\circ(\cdot) \in X^\circ(\cdot, t_*, x_*, \nu(\cdot))$

$$\rho_M(X(\cdot, t_*, x_*, \nu(\cdot))) = \min_{\Theta_{t_*}} \min_{M_\vartheta} \omega(\vartheta, x^\circ(\vartheta), m)$$

For each control $\eta(\cdot) \in \{H(m(\cdot)), [t_*, \vartheta_0]\}$. We introduce the set $\Theta(t_*, x_*, \eta(\cdot))$ of all instants $\vartheta^\circ$ which yield

$$\min_{\Theta_{t_*}} \min_{M_\vartheta} \omega(\vartheta, \varphi(\vartheta, t_*, x_*, \eta(\cdot)), m)$$

Here $\varphi(\cdot, t_*, x_*, \eta(\cdot))$ is the program motion from position $(t_*, x_*)$, generated by control $\eta(\cdot)$. In addition, let

$$\Theta(t_*, x_*, \nu(\cdot)) = \bigcup_{\{\Pi(\nu(\cdot)), [t_*, \vartheta_0] \mid t_*, x_*\}_0} \Theta(t_*, x_*, \eta(\cdot))$$

$$\Theta(t_*, x_*) = \bigcup_{\sigma(t_*, x_*)} \Theta(t_*, x_*, \nu(\cdot))$$

$$M^\circ(\eta(\cdot), \vartheta, t_*, x_*) = \{m^\circ : m^\circ \in M_\vartheta, \min_{M_\vartheta} \omega(\vartheta,$$
$$\varphi(\vartheta, t_*, x_*, \eta(\cdot)), m) = \omega(\vartheta, \varphi, (\vartheta, t_*, x_*, \eta(\cdot)), m^\circ)\}$$

Then for every position $(\omega_0 < \varepsilon_0(t_*, x_*) < \omega^\circ)$ and control $\nu_0(\cdot) \in \Sigma(t_*, x_*)$ we denote by $S_0(t_*, x_*, \nu_0(\cdot))$ the set of all vectors $s_0$ for which

$$s_0' = \left[\frac{\partial}{\partial x} \omega(\vartheta^\circ, \varphi(\vartheta^\circ, t_*, x_*, \eta_0(\cdot)), m_0)\right] S(\vartheta^\circ, t_*, \varphi_0(\cdot), \eta_0(\cdot))$$

where $S(\vartheta, t, \varphi_0(\cdot), \eta_0(\cdot))$ is the fundamental solution matrix [3, 7] for the variational equation corresponding to the control $\eta_0(\cdot)$ and to the program motion

$$\varphi_0(\cdot) = \varphi(\cdot, t_*, x_*, \eta_0(\cdot))$$
$$\eta_0(\cdot) \in \{\Pi(\nu_0(\cdot)), [t_*, \vartheta_0] \mid t_*, x_*\}_0, \ \vartheta^\circ \in \Theta(t_*, x_*, \eta_0(\cdot))$$
$$m_0 \in M^\circ(\eta_0(\cdot), \vartheta^\circ, t_*, x_*)$$

We also introduce the set

$$S_0(t_*, x_*) = \bigcup_{\Sigma(t_*, x_*)} S_0(t_*, x_*, \nu(\cdot))$$

The control optimal in program necessarily satisfies the following condition which expresses Pontriagin's maximum principle [6] in the given program problem.

Theorem 2.1. Let $\rho_M(X(\cdot, t_*, x_*, \nu(\cdot))) \in (\omega_0, \omega^\circ)$. Then for every control $\eta_0(\cdot) \in \{\Pi(\nu(\cdot)), [t_*, \vartheta_0] \mid t_*, x_*\}_0$, for the instant $\vartheta^\circ \in \Theta(t_*, x_*, \eta_0(\cdot))$ and for the point $m_0 \in M^\circ(\eta_0(\cdot), \vartheta^\circ, t_*, x_*)$ the equality

$$\int_\Delta \int_P \int_Q s_0'(t) f(t, \varphi_0(t), u, v)\eta_0(dt \times du \times dv) = \int_\Delta \int_Q \min_P [s_0'(t) f(t, \varphi_0(t), u, v)]\nu(dt \times dv)$$

is fulfilled on every set $\Delta \in \sigma_{[t_*, \vartheta^\circ]}$. Here

$$s_0'(t) = \left[\frac{\partial}{\partial x}\omega(\vartheta^\circ, \varphi_0(\vartheta^\circ), m_0)\right]' S(\vartheta^\circ, t, \varphi_0(\cdot), \eta_0(\cdot))$$

$$\varphi_0(t) = \varphi(t, t_*, x_*, \eta_0(\cdot))$$

We say that a control $\nu_0(\cdot) \in \Sigma(t_*, x_*)$ is regular if it satisfies the following conditions:

1) The set $\Theta(t_*, x_*, \nu_0(\cdot))$ consists of the single point $\vartheta^\circ = \vartheta^\circ(t_*, x_*, \nu_0(\cdot))$.

2) Every control $\eta^{\circ\circ}(\cdot) \in \{\Pi(\nu_0(\cdot)), [t_*, \vartheta_0] \mid t_*, x_*\}_0$ coincides on Borel subsets of the product $[t_*, \vartheta^\circ] \times P \times Q$ with some program control $\eta_0(\cdot) \in \{\Pi(\nu_0(\cdot)), [t_*, \vartheta^\circ]\}$, where $\{\Pi(\nu_0(\cdot)), [t_*, \vartheta^\circ]\}$ is the program of the segment $[t_*, \vartheta^\circ]$, corresponding to the control $\nu_0(\cdot)$ [7].

3) The set $M^\circ(\eta_0(\cdot), \vartheta^\circ, t_*, x_*)$ consists of the single point $m_0$.

Theorem 2.2. Let $\varepsilon_0(t_*, x_*) \in (\omega_0, \omega^\circ)$ and let the control $\nu_0(\cdot) \in \Sigma(t_*, x_*)$ be regular. Then every control $\eta^{\circ\circ}(\cdot) \in \{\Pi(\nu_0(\cdot)), [t_*, \vartheta_0] \mid t_*, x_*\}_0$, solving (2.1) necessarily satisfies the following maximin condition:

$$\int_\Delta \int_P \int_Q s_0'(t) f(t, \varphi^{\circ\circ}(t), u, v)\eta^{\circ\circ}(dt \times du \times dv) =$$
$$\int_\Delta \max_Q \min_P [s_0'(t) f(t, \varphi^{\circ\circ}(t), u, v)] m(dt)$$

Here

$$\varphi^{\circ\circ}(t) = \varphi(t, t_*, x_*, \eta^{\circ\circ}(\cdot))$$

$$s_0'(t) = \left[\frac{\partial}{\partial x}\omega(\upsilon^\circ, \varphi^{\circ\circ}(\upsilon^\circ), m^{\circ\circ})\right]' S(\vartheta^\circ, t, \varphi^{\circ\circ}(\cdot), \eta^{\circ\circ}(\cdot))$$

$$m^{\circ\circ} \in M^\circ(\eta^{\circ\circ}(\cdot), \vartheta^\circ, t_*, x_*), \qquad \vartheta^\circ = \vartheta^\circ(t_*, x_*, \nu_0(\cdot))$$

($\Delta$ is any Borel subset of the interval $[t_*, \vartheta^\circ]$).

The proof is carried out by a scheme analogous to the one in [7].

Using the properties of program motions we can show that the function $\varepsilon_0(t, x)$ is right-continuous at each position $(t_*, x_*)$

$$t_* \in [t_0, \vartheta_0) \setminus \Theta(t_*, x_*) \tag{2.2}$$

while the sets $\Theta(t_*, x_*, \nu(\cdot))$ $(\nu(\cdot) \in \{E(m(\cdot)), [t_*, \vartheta_0]\})$ and $\Theta(t_*, x_*)$ are closed. In addition, the sets $\Sigma(t, x)$ are weakly upper-semicontinuous by inclusion from the right at each position $(t_*, x_*)$ satisfying (2.2).

3. We implement the following auxiliary constructions. Let $(t, x)$ and $(t_*, x_*)$ be two positions $(t \geqslant t_*)$ and $\xi(\cdot)$ be the probability over Borel subsets $Q$, $\nu^\circ(\cdot) \in \Sigma(t, x)$ and $\nu_\xi^\circ(\cdot)$ obtained by splicing with probability $\xi(\cdot)$ by extending the constant control $\xi(\cdot)$ over the half-interval $[t_*, t)$ of the instantaneous control $\nu_t^\circ{}_{,}(\cdot)$. In the program $\{\Pi(\nu_\xi^\circ(\cdot)), [t_*, \vartheta_0]\}$ we select any control $\eta_\xi^\circ(\cdot)$ optimal for the position $(t_*, x_*)$, while in the set $\Theta(t_*, x_*, \eta_\xi^\circ(\cdot))$ we select any point $\vartheta_\xi^\circ$. Next, from the set $M^\circ(\eta_\xi^\circ(\cdot), \vartheta_\xi^\circ, t_*, x_*)$ we choose any element $m_\xi^\circ$. By $O_\delta(t_*, x_*)$ we denote the right $\delta$-semineighborhood of position $(t_*, x_*)$: $0 \leqslant t - t_* < \delta$, $\|x - x_*\| < \delta$.

Lemma 3.1. For any position $(t_*, x_*)$, $t_* \in [t_0, \vartheta_0)$ and any number $\alpha > 0$ there exists $\delta > 0$ such that for an arbitrary choice of position $(t, x) \in O_\delta(t_*, x_*)$ the controls $\nu^\circ(\cdot) \in \Sigma(t, x)$, $\xi(\cdot)$ and $\eta_\xi^\circ(\cdot) \in \{\Pi(\nu_\xi^\circ(\cdot)), [t_*, \vartheta_0] \mid t_*, x_*\}_0$

$$\Theta(t_*, x_*, r_\xi^\circ(\cdot)) \subset \Theta^\alpha(t_*, x_*)$$

The proof relies on the weak upper-semicontinuity by inclusion of the sets $\Sigma(t, x)$.

By virtue of the closedness of set $\Theta(t_*, x_*)$ and of Lemma 3.1, for every position $(t_*, x_*)$, $t_* \in [t_0, \vartheta_0) \setminus \Theta(t_*, x_*)$ there exists $\delta > 0$ such that for an arbitrary choice of $\nu^\circ(\cdot)$, $\xi(\cdot)$, $\eta_\xi^\circ(\cdot)$ from the appropriate sets

$$\Theta(t_*, x_*, \eta_\xi^\circ(\cdot)) \subset \Theta_t$$

for every position from $O_\delta(t_*, x_*)$. Below we assume that the adjacent position $(t, x)$ is selected from this condition. The control from $\{\Pi(\nu^\circ(\cdot)), [t, \vartheta_0]\}$ coinciding with $\eta_\xi^\circ(\cdot)$ on $[t, \vartheta_0] \times P \times Q$ will be denoted by $\bar{\eta}_\xi^\circ(\cdot)$. Then we can show that for every position $(t_*, x_*)$, $t_* \in [t_0, \vartheta_0) \setminus \Theta(t_*, x_*)$, we can find, for any $\alpha > 0$, a $\delta > 0$ such that for every position $(t, x) \in O_\delta(t_*, x_*)$

$$|\omega(\vartheta_\xi^\circ, \bar{\varphi}_\xi^\circ(\vartheta_\xi^\circ), m_\xi^\circ) - \varepsilon_0(t_*, x_*)| < \alpha$$

where

$$\bar{\varphi}_\xi^\circ(\vartheta_\xi^\circ) = \varphi(\vartheta_\xi^\circ, t, x, \bar{\eta}_\xi^\circ(\cdot))$$

for an arbitrary choice of $\nu^\circ(\cdot)$, $\xi(\cdot)$, $\eta_\xi^\circ(\cdot)$, $\vartheta_\xi^\circ$ and $m_\xi^\circ$ from the appropriate sets.

With due regard to this, for every position $(t_*, x_*)$ satisfying the condition

$$\varepsilon_0(t_*, x_*) \in (\omega_0, \omega^\circ), \qquad t_* \in [t_0, \vartheta_0) \setminus \Theta(t_*, x_*) \tag{3.1}$$

and for any position $(t, x)$ from a sufficiently small right $\delta$-semineighborhood of $(t_*, x_*)$, for each $\nu^\circ(\cdot) \in \Sigma(t, x)$ and $\xi(\cdot)$ we define the set $S_*(t, x \mid t_*, x_*, \nu^\circ(\cdot), \xi(\cdot))$ consisting of all vectors $s$ such that

$$s' = \left[\frac{\partial}{\partial x}\omega(\vartheta_\xi^\circ, \bar{\varphi}_\xi^\circ(\vartheta_\xi^\circ), m_\xi^\circ)\right]^* S(\vartheta_\xi^\circ, t, \bar{\varphi}_\xi^\circ(\cdot), \bar{\eta}_\xi^\circ(\cdot)) \tag{3.2}$$

where

$$\eta_\xi^\circ(\cdot) \in \{\Pi(\nu_\xi^\circ(\cdot)), [t_*, \vartheta_0] \mid t_*, x_*\}_0$$
$$\vartheta_\xi^\circ \in \Theta(t_*, x_*, \eta_\xi^\circ(\cdot)), \quad m_\xi^\circ \in M^\circ(\eta_\xi^\circ(\cdot), \vartheta_\xi^\circ, t_*, x_*)$$

Lemma 3.2. For every position $(t_*, x_*)$ satisfying (3.1) and for any number $\alpha > 0$ we can find $\delta > 0$ such that for each position $(t, x) \in O_\delta(t_*, x_*)$ there exists, for any control $\nu^\circ(\cdot) \in \Sigma(t, x)$, a control $\nu_0(\cdot) \in \Sigma(t_*, x_*)$ for which

$$\bigcup_{\{\xi(\cdot)\}_Q} S_*(t, x\,|\,t_*, x_*, \nu^\circ(\cdot), \xi(\cdot)) = \tag{3.3}$$
$$S_*(t, x\,|\,t_*, x_*, \nu^\circ(\cdot)) \subset S_0^{\,\alpha}(t_*, x_*, \nu_0(\cdot))$$

where $S^\alpha$ is the $\alpha$-neighborhood of set $S$ in the Euclidean metric $\|\cdot\|$, while $\{\xi(\cdot)\}_Q$ is the collection of all probability measures on $Q$.

Below we assume the fulfillment of the following condition.

Condition A. For every position $(t_*, x_*)$ satisfying (3.1) and for any control $\nu_0(\cdot) \in \Sigma(t_*, x_*)$ there exists a vector $v_0 \in Q$ for which the equality

$$\min_P s_0' f(t_*, x_*, u, v_0) = \max_Q \min_P s_0' f(t_*, x_*, u, v)$$

is fulfilled on every vector $s_0 \in S_0(t_*, x_*, \nu_0(\cdot))$.

Theorem 3.1. For every position $(t_*, x_*)$ satisfying (3.1), with respect to any number $\gamma > 0$ we can find $\delta > 0$ such that for any position $(t, x) \in O_\delta(t_*, x_*)$

$$\varepsilon_0(t, x) - \varepsilon_0(t_*, x_*) \leqslant \max_{S_0(t_*, x_*)} [s'(x - x_*) - \tag{3.4}$$
$$\max_Q \min_P s' f(t_*, x_*, u, v)(t - t_*)] + \gamma \max(t - t_*, \|x - x_*\|)$$

Proof. Let $(t_*, x_*)$ satisfy the lemma's conditions and $\alpha$ be any positive number. We assume that the adjacent position $(t, x)$ is chosen from such a neighborhood of $(t_*, x_*)$ that (3.3) is fulfilled (such a neighborhood exists by virtue of Lemma 3.2). On the other hand

$$\varepsilon_0(t, x) - \varepsilon_0(t_*, x_*) \leqslant \omega(\vartheta_\xi^\circ, \bar{\varphi}_\xi^\circ(\vartheta_\xi^\circ), m_\xi^\circ) - \omega(\vartheta_\xi^\circ, \varphi_\xi^\circ(\vartheta_\xi^\circ), m_\xi^\circ) \tag{3.5}$$

for any $\nu^\circ(\cdot) \in \Sigma(t, x)$, $\xi(\cdot)$, $\eta_\xi^\circ(\cdot) \in \{\Pi(\nu_\xi^\circ(\cdot)), [t_*, \vartheta_0] \mid t_*, x_*\}_0$, $\vartheta_\xi^\circ \in \Theta(t_*, x_*, \eta_\xi^\circ(\cdot))$ and $m_\xi^\circ \in M^\circ(\eta_\xi^\circ(\cdot), \vartheta_\xi^\circ, t_*, x_*)$. Then, having chosen any control $\nu^\circ(\cdot) \in \Sigma(t, x)$, we choose a control $\nu_0(\cdot) \in \Sigma(t_*, x_*)$ such that (3.3) is fulfilled, after which, with due regard to Condition A we select a probability $\xi(\cdot)$ such that the equality

$$\int_Q \min_P [s_0' f(t_*, x_*, u, v)]\, \xi(dv) = \max_Q \min_P s_0' f(t_*, x_*, u, v)$$

is fulfilled on any vector $s_0 \in S_0(t_*, x_*, \nu_0(\cdot))$. We use the indicated $\nu^\circ(\cdot)$ and $\xi(\cdot)$ in estimate (3.5). Subsequent derivation is carried out allowing for this estimate and for the differentiability of the function $\omega(\cdot)$ with respect to $x$ as in [8].

**4.** Let $W_\varepsilon$ be the set of all positions $(t, x)$, $t \in [t_0, \vartheta_0]$, for which $\varepsilon_0(t, x) \leqslant \varepsilon$. This set is closed for every $\varepsilon$. We say that a probability $\mu(\cdot)$ on $P \times Q$ is consistent with the probability $\xi(\cdot)$ on $Q$ if $\mu(P \times B) = \xi(B)$ for each Borel subset $B \subset Q$. (By a probability we mean a normed measure on a $\sigma$-algebra of Borel subsets of the corresponding space).

Condition B. For every position $(t_*, x_*)$ satisfying (3.1) and for any probability $\xi(\cdot)$ on $Q$ there exists a probability $\mu(\cdot)$ on $P \times Q$, consistent with $\xi(\cdot)$, such that

$$s_0' \int_P \int_Q f(t_*, x_*, u, v)\, \mu(du \times dv) \leqslant \max_Q \min_P s_0' f(t_*, x_*, u, v)$$

uniformly with respect to $s_0 \in S_0(t_*, x_*)$.

Allowing for Theorem 3.1, the following theorem is proved.

Theorem 4.1. Let Conditions A, B be fulfilled. Then the sets $W_\varepsilon$ are $u$-stable for every $\varepsilon \in [\omega_0, \omega^\circ)$: for every position $(t_*, x_*) \in W_\varepsilon$, for the probability $\xi(\cdot)$

on $Q$ and for an instant $t^* \in [t_*, \vartheta_0]$, in the family of all possible program motions on $[t_*, t^*]$, generated by controls from the program $\{\Pi(\nu^{(\xi)}(\cdot)), [t_*, t^*]\}$, we can find either a motion $\varphi^\circ(t)$ for which

$$\min_{\Theta \cap [t_*, t^*]} \min_{M_\vartheta} \omega(\vartheta, \varphi^\circ(\vartheta), m) \leqslant \varepsilon$$

or a motion $\varphi_0(t)$ for which the position $(t, \varphi_0(t)) \in W_\varepsilon$ for all $t \in [t_*, t^*]$. Here $\nu^{(\xi)}(\cdot)$ is a control from class $\{E(m(\cdot)), [t_*, t^*]\}$ [8] such that the instantaneous control $\nu_t^{(\xi)}(\cdot)$ corresponding to it is the probability $\xi(\cdot)$ for almost all $t \in [t_*, t^*]$.

To obtain the necessary conditions for the $u$-stability of sets $W_\varepsilon$ $(\varepsilon \in [\omega_0, \omega^\circ))$ we implement the following auxiliary constructions. Once again let $(t_*, x_*)$ and $(t, x)$ be such that $t_* \in [t_0, \vartheta_0)$ and $t \geqslant t_*$. Further, let $\nu_0(\cdot) \in \Sigma(t_*, x_*)$, let $\bar{\nu}_0(\cdot) \in \{E(m(\cdot)), [t, \vartheta_0]\}$ and let it coincide with $\nu_0(\cdot)$ on $[t, \vartheta_0] \times Q$, and let

$$\bar{\eta}_0(\cdot) \in \{\Pi(\bar{\nu}_0(\cdot)), [t, \vartheta_0] \mid t, x\}_0$$
$$\bar{\vartheta}^\circ \in \Theta(t, x, \bar{\eta}_0(\cdot)), \quad \bar{m}_0 \in M^\circ(\bar{\eta}_0(\cdot), \bar{\vartheta}^\circ, t, x)$$
$$\eta_0(\cdot) \in \{\Pi(\nu_0(\cdot)), [t_*, \vartheta_0]\}$$

where the values of measures $\eta_0(\cdot)$ and $\bar{\eta}_0(\cdot)$ coincide on the Borel subsets of $[t, \vartheta_0] \times P \times Q$. Then

$$\varepsilon_0(t, x) - \varepsilon_0(t_*, x_*) \geqslant \omega(\bar{\vartheta}^\circ, \bar{\varphi}_0(\bar{\vartheta}^\circ), \bar{m}_0) - \omega(\bar{\vartheta}^\circ, \varphi_0(\bar{\vartheta}^\circ), \bar{m}_0) \quad (4.1)$$
$$\bar{\varphi}_0(\cdot) = \varphi(\cdot, t_*, x_*, \bar{\eta}_0(\cdot)), \qquad \bar{\varphi}_0(\cdot) = \varphi(\cdot, t, x, \eta_0(\cdot))$$

We can show that for every position $(t_*, x_*)$, $t_* \in [t_0, \vartheta_0) \setminus \Theta(t_*, x_*)$, for any $\alpha > 0$ we can find $\delta > 0$ such that for any neighboring position $(t, x) \in O_\delta(t_*, x_*)$

$$| \omega(\bar{\vartheta}^\circ, \bar{\varphi}_0(\bar{\vartheta}^\circ), \bar{m}_0) - \varepsilon_0(t_*, x_*) | < \alpha$$

for an arbitrary choice of $\nu_0(\cdot)$, $\bar{\eta}_0(\cdot)$, $\bar{\vartheta}^\circ$ and $\bar{m}_0$ from the appropriate sets. Therefore, for every position $(t_*, x_*)$ satisfying (3.1) and for any adjacent position $(t, x)$ from a sufficiently small right $\delta$-semineighborhood of $(t_*, x_*)$ we can determine, for each control $\nu_0(\cdot) \in \Sigma(t_*, x_*)$, the set $S^*(t, x \mid t_*, x_*, \nu_0(\cdot))$ of all vectors $s$

$$s' = \left[\frac{\partial}{\partial x} \omega(\bar{\vartheta}^\circ, \bar{\varphi}_0(\bar{\vartheta}^\circ), \bar{m}_0)\right]' S(\bar{\vartheta}^\circ, t, \bar{\varphi}_0(\cdot), \bar{\eta}_0(\cdot))$$

Lemma 4.1. For any position $(t_*, x_*)$ satisfying (3.1) and any control $\nu_0(\cdot) \in \Sigma(t_*, x_*)$, for every $\alpha > 0$ we can find $\delta > 0$ such that

$$S^*(t, x \mid t_*, x_*, \nu_0(\cdot)) \subset S_0^\alpha(t_*, x_*, \nu_0(\cdot))$$

for each position $(t, x) \in O_\delta(t_*, x_*)$.

Theorem 4.2. Let the set $W_\varepsilon$ be $u$-stable for every $\varepsilon \in [\omega_0, \omega^\circ)$. Then for each position $(t_*, x_*)$ satisfying (3.1) and for any probability $\xi(\cdot)$ on $Q$ there exists a probability $\mu(\cdot)$ on $P \times Q$, consistent with $\xi(\cdot)$, such that

$$\min_{S_0(t_*, x_*, \nu_0(\cdot))} \Big[ s_0' \int_P \int_Q f(t_*, x_*, u, v) \times \quad (4.2)$$
$$\mu(du \times dv) - \max_Q \min_P s_0' f(t_*, x_*, u, v) \Big] \leqslant 0$$

for each control $\nu_0(\cdot) \in \Sigma(t_*, x_*)$.

Plan of the proof. For every position satisfying the lemma's conditions there exists an instant $\tau^* > t_*$ such that for every preselected probability $\xi(\cdot)$ the inequality

$$\min_{\Theta\cap[t_*,\tau^*]} \min_{M_\vartheta} \omega(\vartheta, \varphi, \vartheta(\vartheta, t_*, x_*, \eta(\cdot)), m) > \varepsilon_0(t_*, x_*)$$

is fulfilled for any program motion $\varphi(t, t_*, x_*, \eta(\cdot))$ for which $\eta_t(P \times B) = \xi(B)$ for any Borel subsets of $Q$. By the definition of $u$-stability we conclude that for each probability $\xi(\cdot)$ there must exist a control $\eta^*(\cdot)$, consistent with $\xi(\cdot)$, such that

$$\varepsilon_0(t, \varphi(t, t_*, x_*, \eta^*(\cdot))) \leqslant \varepsilon_0(t_*, x_*) \quad \text{for all } t \in [t_*, \tau^*]$$

Assume that the theorem is incorrect. Then, with due regard to what we have said above, at the position $(t_*, x_*)$ where (4.2) is violated for some $\xi(\cdot)$ and $\nu_0(\cdot)$, for some sequence $\{\tau_n\}$ converging to $t_*$ from the right $(\tau_n > t_*)$, we can use estimate (4.1) just under that control $\nu_0(\cdot)$ by which condition (4.2) is violated for a preselected $\xi(\cdot)$. But then, allowing for the differentiability of function $\omega(\cdot)$ with respect to $x$ and for Lemma 4.1, for sufficiently large $n$ we obtain

$$\varepsilon_0(\tau_n, x_n) > \varepsilon_0(t_*, x_*), \qquad x_n = \varphi(\tau_n, t_*, x_*, \eta^*(\cdot))$$

Corollary. Suppose that under each control $\nu_0(\cdot) \in \Sigma(t_*, x_*)$ the set $S_0(t_*, x_*, \nu_0(\cdot))$ consists of the single vector $s_0 = s_0(t_*, x_*, \nu_0(\cdot))$ for every position $(t_*, x_*)$ satisfying (3.1). The Condition B is necessary and sufficient for the sets $W_\varepsilon$ to be $u$-stable for any $\varepsilon \in [\omega_0, \omega^\circ)$.

5. Let $U^e$ be the strategy extremal [2] to set $W_\varepsilon$ and let $U_v{}^e$ be the counterstrategy [8] extremal to that same set.

Theorem 5.1. Let $\varepsilon = \varepsilon_0(t_0, x_0) \in [\omega_0, \omega^\circ)$ and let Conditions A, B be fulfilled. Then, under the condition that a saddle point with respect to $(u, v)$ exists in the small game [2], the strategy $U^\circ = U^e$ extremal to set $W_\varepsilon$ solves Problem 1 by guaranteeing the fulfillment of (1.1).

Theorem 5.2. Let $\varepsilon = \varepsilon_0(t_0, x_0) \in [\omega_0, \omega^\circ)$ and let Conditions A, B be fulfilled. Then the counterstrategy $U_v{}^\circ = U_v{}^e$ extremal to set $W_\varepsilon$ solves Problem 1 by guaranteeing here the fulfillment of (1.2).

For the control $\nu_0(\cdot) \in \Sigma(t_0, x_0)$ we form the set $W(\nu_0(\cdot))$ of all positions $(t, w)$

$$w = \varphi(t, t_0, x_0, \eta(\cdot)), \qquad \eta(\cdot) \in \{\Pi(\nu_0(\cdot)), [t_0, \vartheta_0]\}$$

Let $V^e$ be the second player's strategy [8], extremal [2] to set $W(\nu_0(\cdot))$.

Theorem 5.3. Strategy $V^e$ ensures the solution of Problem 3 for any $\varepsilon \leqslant \varepsilon_0(t_0, x_0)$.

Plan of the proof. Let $x_{\Delta(i)}[t]$ be an Euler polygonal line corresponding to the strategy $V^e$ and let $\tau_k{}^{(i)} = t_*$ be a node of the partitioning $\Delta^{(i)}$, and

$$x_* = x_{\Delta(i)}[t_*] \overline{\in} W_{t_*}(\nu_0(\cdot))$$

$$W_{t_*}(\nu_0(\cdot)) = \{w: (t_*, w) \in W(\nu_0(\cdot))\}$$

In addition, let $v^e = v[t_*]$, $u[t]$ be the control realizing the given Euler polygonal line, $s$ be the vector $w^\circ - x_*$, where $w^\circ$ is a point of set $W_{t_*}(\nu_0(\cdot))$ closest to $x_*$ in the

Euclidean metric and

$$\min_P s'f(t_*, x_*, u, v^e) = \max_Q \min_R s'f(t_*, x_*, u, v)$$

Then, in the program $\{\Pi(\nu_0(\cdot)), [\tau_k^{(i)}, \tau_{k+1}^{(i)}]\}$ we can find a control $\eta^*(\cdot)$ such that

$$s' \int_{t_*}^{t} \int_P \int_Q f(t_*, x_*, u, v)\, \eta^*(d\tau \times du \times dv) = \int_{t_*}^{t} \int_Q \min_P [s'f(t_*, x_*, u, v)]\, \nu_0(d\tau \times dv)$$

for every $t \in [\tau_k^{(i)}, \tau_{k+1}^{(i)}]$. Hence, with due regard to the inequalities

$$s' \int_{t_*}^{t} f(t_*, x_*, u[\tau], v^e)\, m(d\tau) \geqslant \int_{t_*}^{t} \max_Q \min_P [s'f(t_*, x_*, u, v)]\, m(d\tau) \geqslant$$
$$\int_{t_*}^{t} \int_Q \min_P [s'f(t_*, x_*, u, v)]\, \nu_0(d\tau \times dv)$$

we derive a local estimate analogous to the one used in [4]. From this estimate, in analogy with [4], we derive the barrier properties of strategy $V^e$.

Theorem 5.4. Let $\varepsilon = \varepsilon_0(t_0, x_0) \in [\omega_0, \omega^\circ)$ and let Conditions A, B and the small game saddle point condition be fulfilled. Then the pair of strategies $(U^\circ = U^e, V^\circ = V^e)$ solves Problem 2. Here $\varepsilon = \varepsilon_0(t_0, x_0)$ is the value of the game in pure strategies.

Problems 1–3 admit of an intuitive representation when $M$ is a closed subset of $\Theta \times R^n$, and $\omega(\vartheta, x, m) = \|x - m\|$. The possible noncompactness of $M$ is unessential here since the problem reduces to an encounter-evasion problem with some compact subset of $M$.

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